Properties of limits (P59)

$$\lim_{x\to c} f(x) = L \qquad \lim_{x\to c} g(x) = k$$

Scalar multiple: $\lim_{x\to c}[b\,f(x)] = bL$

Sum/Difference: $\lim_{x\to c}[f(x) \pm g(x)] = L \pm K$

Product: $\lim_{x\to c}[f(x) \cdot g(x)] = L \cdot K$

Quotient: $\lim_{x\to c}[f(x)/g(x)] = L/K \quad (\mathrm{K} \neq 0)$

Power: $\lim_{x\to c}[f(x)]^n = L^n \qquad (n \in Z^+)$

Radical: $\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{L} \qquad (n \in Z^+)$

Composite Function: $\lim_{x\to c} g(x) = L\,, and\ \lim_{x\to L} f(x) = f(L)$

$$\lim_{x\to c} f(g(x)) = f(\lim_{x\to c} g(x)) = f(L)$$

Example 1: Prove $\lim_{x\to c}[f(x) + g(x)] = L + K$

Proof: For each $\varepsilon > 0$, to prove $|[f(x) + g(x)] - (L + K)| < \varepsilon$

$\Leftarrow |[f(x) - L] + [g(x) - K]| < \varepsilon$

$\Leftarrow |f(x) - L| + |g(x) - K| < \varepsilon$

$\Leftarrow |f(x) - L| < \frac{\varepsilon}{2}$ and $|g(x) - K| < \frac{\varepsilon}{2}$

There exist $\delta_1 > 0$ and $\delta_2 > 0$, so that $|f(x) - L)| < \varepsilon_1 = \frac{\varepsilon}{2}$ and $|g(x) - k)| < \varepsilon_2 = \frac{\varepsilon}{2}$. Let $\delta = \{\delta_1, \delta_2\}_{min}$>0, we have $|f(x) - L)| + |g(x) - K| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

Example 2: Prove $\lim_{x\to c}[f(x) \cdot g(x)] = L \cdot K$

Proof: $h(x) = f(x) \cdot g(x),\ Limit = L \cdot K$

$$\left.\begin{array}{l}\exists\delta_1, so\ that\ |f(x)-L|<\varepsilon_1 \\ \exists\delta_2, so\ that\ |g(x)-K|<\varepsilon_2\end{array}\right\} \Rightarrow |[f(x) - L][g(x) - K]| < \varepsilon_1\varepsilon_2$$

$$\begin{aligned}&[f(x) - L] \cdot [g(x) - K] \\ &\qquad = f(x)g(x) - Lg(x) - Kf(x) + LK \\ &\qquad = [f(x)g(x) - LK] + [LK - Lg(x)] + [LK - Kf(x)]\end{aligned}$$

$\therefore f(x)g(x) - LK = [f(x) - L] \cdot [g(x) - K] - L[K - g(x)] - K[L - f(x)]$

For each to $\varepsilon > 0$, to prove $[f(x) \cdot g(x) - LK] < \varepsilon$

$\Leftarrow |[f(x) - L] \cdot [g(x) - K] - L[K - g(x)] - K[L - f(x)]| < \varepsilon$

$\Leftarrow |\varepsilon_1\varepsilon_2 - L\varepsilon_2 - K\varepsilon_1| < \varepsilon$

$\Leftarrow -\varepsilon < \varepsilon_1\varepsilon_2 - L\varepsilon_2 - K\varepsilon_1 < \varepsilon$ (Let $\varepsilon_1 = \varepsilon_2 = x$)

$\Leftarrow -\varepsilon < x^2 - (L+K)x < \varepsilon$

$\Leftarrow x^2 - (L+K)x - \varepsilon < 0$

$\Leftarrow \frac{1}{2}\left[(L+K) - \sqrt{(L+K)^2 + 4\varepsilon}\right] < x < \frac{1}{2}\left[(L+K) + \sqrt{(L+K)^2 + 4\varepsilon}\right]$

There exist $\delta_1 > 0$ and $\delta_2 > 0$, so that $\varepsilon_1 = \varepsilon_2 = x$

Example 3: The limit of a polynomial

$$\lim_{x\to 2}(4x^2+3) = \lim_{x\to 2} 4x^2 + \lim_{x\to 2} 3 = 4\lim_{x\to 2} x^2 + \lim_{x\to 2} 3 = 4(2^3) + 3 = 19$$

Example 4: The limit of a rational function

$$\lim_{x\to 2}\frac{2x^2+1}{x-1} = \frac{\lim_{x\to 2}(2x^2)+1}{\lim_{x\to 2}(x-1)} = \frac{8+1}{2-1} = 9$$

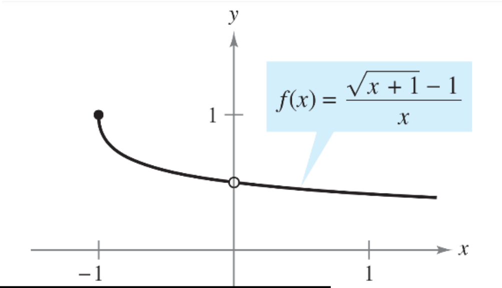
Dividing Out and Rationalizing (P63)

$\lim_{x\to 0}\frac{\sqrt{x+1}-1}{x}$ is an **indeterminate form** $\frac{0}{0}$ ($\frac{0}{0}$ 型不定式)

Solution: (Rationalize the numerator 分子有理化)

$\lim_{x\to 0}\frac{\sqrt{x+1}-1}{x} = \lim_{x\to 0}\frac{(\sqrt{x+1}-1)(\sqrt{x+1}+1)}{x(\sqrt{x+1}+1)} =$

$\lim_{x\to 0}\frac{(x+1)-1}{x(\sqrt{x+1}+1)} = \lim_{x\to 0}\frac{1}{\sqrt{x+1}+1} = \frac{1}{1+1} = \frac{1}{2}$



Seven **indeterminate forms**: $\frac{0}{0}, \frac{\infty}{\infty}, 0\times\infty, \infty-\infty, 0^0, 1^\infty$ *and* ∞^0

Example 1: Find the limit $\lim_{\Delta x\to 0}\frac{\sqrt{x+\Delta x}-\sqrt{x}}{\Delta x}$

Solution: (Rationalize the numerator 分子有理化)

$$\lim_{\Delta x\to 0}\frac{\sqrt{x+\Delta x}-\sqrt{x}}{\Delta x} = \lim_{\Delta x\to 0}\frac{(\sqrt{x+\Delta x}-\sqrt{x})(\sqrt{x+\Delta x}+\sqrt{x})}{\Delta x(\sqrt{x+\Delta x}+\sqrt{x})}$$

$$= \lim_{\Delta x\to 0}\frac{(x+\Delta x)-x}{\Delta x(\sqrt{x+\Delta x}+\sqrt{x})} = \lim_{\Delta x\to 0}\frac{\Delta x}{\Delta x(\sqrt{x+\Delta x}+\sqrt{x})}$$

$$= \lim_{\Delta x\to 0}\frac{1}{\sqrt{x+\Delta x}+\sqrt{x}} = \frac{1}{\sqrt{x}+\sqrt{x}} = \frac{1}{2\sqrt{x}} = \frac{\sqrt{x}}{2x}$$

Example 2: Given $\lim_{x\to 2}\frac{x^2+ax+b}{x-2} = 3$, evaluate a and b.

Solution: $\lim_{x\to 2} x-2 = 0 \Rightarrow \lim_{x\to 2}\frac{x^2+ax+b}{x-2} = 3$ is an indeterminate form $\frac{0}{0}$.

Let $x^2+ax+b = (x-2)(x+k)$

$$\Rightarrow \lim_{x\to 2}\frac{(x-2)(x+k)}{x-2} = 0 \ \Rightarrow \ x+k=3 \ \Rightarrow \ k=1$$

$$\Rightarrow \ (x-2)(x+1) = x^2 - x - 2 = x^2 + ax + b$$

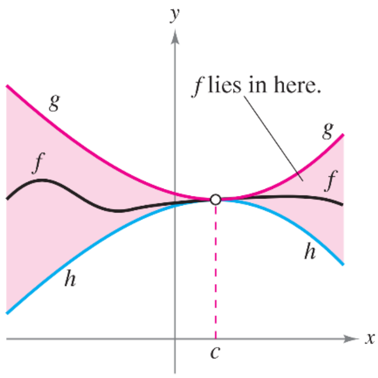
$$\Rightarrow \ a=-1, b=-2$$

The Squeeze Theorem 夹逼定理 (P65)

The Squeeze Theorem: If $h(x) \le f(x) \le g(x)$ for all x in an open interval containing c, except possibly at c itself, and if

$$\lim_{x\to c} h(x) = L = \lim_{x\to c} g(x)$$

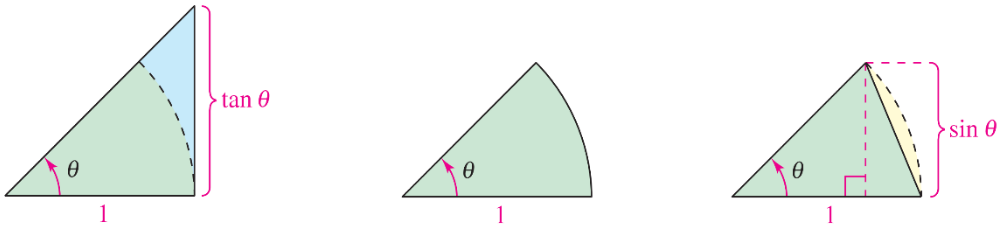
then $\lim_{x\to c} f(x) = L$



Two Special Trigonometric Limits:

1. $\lim_{x\to 0}\frac{\sin x}{x} = 1$

2. $\lim_{x\to 0}\frac{1-\cos x}{x} = 0$



The above figure shows a circular sector squeezed between two triangles. So we can have:

$$\frac{\tan\theta}{2} \ge \frac{\theta}{2} \ge \frac{\sin\theta}{2} \quad \Rightarrow \quad \tan\theta \ge \theta \ge \sin\theta$$

$$\frac{\sin x}{\tan x} \le \frac{\sin x}{x} \le \frac{\sin x}{\sin x} \quad \Rightarrow \quad \frac{\sin x}{\sin x/\cos x} = \cos x \le \frac{\sin x}{x} \le 1$$

$$\lim_{x\to 0}\cos x = 1 \quad \Rightarrow \quad \lim_{x\to 0}\frac{\sin x}{x} = 1$$

Exercise 1: Find the limit (if it exists)

1.1 $\lim_{x \to 0} \frac{\sqrt{x+5}-\sqrt{5}}{x}$

1.2 $\lim_{x \to 0} \frac{[1/(3+x)]-(1/3)}{x}$

1.3 $\lim_{\Delta x \to 0} \frac{(x+\Delta x)^3 - x^3}{\Delta x}$

Exercise 2: For any $\varepsilon > 0$, there exists $\delta > 0$, when $|x-0| < \delta$, there is $\left|\frac{f(x)}{x} - 1\right| < \varepsilon$, find $\lim_{x \to 0} f(x)$.

Exercise 3: Find the limit (if it exists)

3.1 $\lim_{x \to 0} \frac{\sin x}{5x}$

3.2 $\lim_{x \to 0} \frac{\sin^2 x}{x}$

3.3 $\lim_{\theta\to 0}\frac{\cos\theta\tan\theta}{\theta}$

3.4 $\lim_{\theta\to\pi}(\theta\sec\theta)$

3.5 $\lim_{x\to\pi/4}\frac{1-\tan x}{\sin x-\cos x}$

3.6 $\lim_{x\to 0}\frac{\sin 2x}{\sin 3x}$

3.7 $\lim_{x\to 0}\frac{1-\cos x}{x}$

3.8 $\lim_{x\to 0}\frac{\cos x-1}{2x^2}$

3.9 $\lim_{x\to 0}\frac{\sin^2 x}{x}$

3.10 $\lim_{x\to 0}\frac{\sin x}{\sqrt[3]{x}}$

3.11 $\lim_{x\to\pi}\frac{\sin x}{x}$

Exercise 4: indeterminate form

4.1 Given $\lim_{x\to 3}\frac{x^2+ax+b}{x^2-2x-3}=\frac{-1}{4}$, evaluate a and b.

4.2 If $\lim_{x\to 0}\frac{\tan ax}{\sin 3x}=-2$, find a.

Exercise 1: Find the limit (if it exists)

1.1 $\lim_{x\to0}\frac{\sqrt{x+5}-\sqrt{5}}{x} = \lim_{x\to0}\frac{\sqrt{x+5}-\sqrt{5}}{x}\cdot\frac{\sqrt{x+5}+\sqrt{5}}{\sqrt{x+5}+\sqrt{5}} = \lim_{x\to0}\frac{x+5-5}{x(\sqrt{x+5}+\sqrt{5})}$

$= \lim_{x\to0}\frac{1}{\sqrt{x+5}+\sqrt{5}} = \frac{1}{\sqrt{5}+\sqrt{5}} = \frac{1}{2\sqrt{5}} = \frac{\sqrt{5}}{10}$

1.2 $\lim_{x\to0}\frac{[1/(3+x)]-(1/3)}{x} = \lim_{x\to0}\frac{\frac{3-(3+x)}{(3+x)3}}{x} = \lim_{x\to0}\frac{-x}{x(3+x)3}$

$= \lim_{x\to0}\frac{-1}{(3+x)3} = \frac{-1}{9}$

1.3 $\lim_{\Delta x\to0}\frac{(x+\Delta x)^3-x^3}{\Delta x} = \frac{\cancel{x^3}+3x^2\Delta x+3x\Delta x^2+\Delta x^3-\cancel{x^3}}{\Delta x}$

$= \lim_{\Delta x\to0}(3x^2+3x\Delta x+\Delta x^2) = 3x^2$

1.4 $\lim_{x\to0}\frac{\sqrt{2+x}-\sqrt{2}}{x} = \lim_{x\to0}\frac{(\sqrt{2+x}-\sqrt{2})(\sqrt{2+x}+\sqrt{2})}{x}$

$= \lim_{x\to0}\frac{(2+x)-2}{x(\sqrt{2+x}+\sqrt{2})} = \lim_{x\to0}\frac{1}{\sqrt{2+x}+\sqrt{2}} = \frac{1}{\sqrt{2}+\sqrt{2}} = \frac{\sqrt{2}}{4}$

Exercise 2: Given $\lim_{x\to3}\frac{x^2+ax+b}{x^2-2x-3} = \frac{-1}{4}$, evaluate a and b.

$\lim_{x\to3}(x^2-2x-3) = \lim_{x\to3}(x-3)(x+1) = 0 \Rightarrow \lim_{x\to3}\frac{x^2+ax+b}{x^2-2x-3} = \frac{-1}{4}$

is an indeterminate form $0/0$

let $x^2+ax+b = (x-3)(x+k) \Rightarrow k=1 \Rightarrow \begin{cases} a=-7 \\ b=12 \end{cases}$

Exercise 3: Find the limit (if it exists)

3.1 $\lim_{x\to0}\frac{\sin x}{5x} = \lim_{x\to0}\left[\left(\frac{1}{5}\right)\left(\frac{\sin x}{x}\right)\right] = \frac{1}{5}\lim_{x\to0}\frac{\sin x}{x} = \frac{1}{5}$

3.2 $\lim_{x\to0}\frac{\sin^2 x}{x} = \lim_{x\to0}\sin x\cdot\lim_{x\to0}\frac{\sin x}{x} = 0\cdot1 = 0$

3.3 $\lim_{\theta\to0}\frac{\cos\theta\tan\theta}{\theta} = \lim_{\theta\to0}\frac{\cos\theta\frac{\sin\theta}{\cos\theta}}{\theta} = \lim_{\theta\to0}\frac{\sin\theta}{\theta} = 1$

3.4 $\lim_{\theta\to\pi}(\theta\sec\theta) = \lim_{\theta\to\pi}\left(\theta\frac{1}{\cos\theta}\right) = \left(\lim_{\theta\to\pi}\theta\right)\left(\lim_{\theta\to\pi}\frac{1}{\cos\theta}\right) = \pi(-1) = -\pi$

3.5 $\lim_{x\to\pi/4}\frac{1-\tan x}{\sin x-\cos x} = \lim_{x\to\frac{\pi}{4}}\frac{1-\frac{\sin x}{\cos x}}{\sin x-\cos x} = \lim_{x\to\frac{\pi}{4}}\frac{\cos x-\sin x}{\sin x-\cos x} = -1$

3.6 $\lim_{x\to0}\frac{\sin 2x}{\sin 3x}$

方法1：设 $\frac{\sin 2x}{\sin 3x} = t$ $\therefore \frac{1}{t} = \frac{\sin 3x}{\sin 2x} = \frac{\sin 2x\cos x+\cos 2x\sin x}{\sin 2x}$

$\frac{1}{t} = \cos x + \frac{\cos 2x\sin x}{\sin 2x} = \cos x + \frac{\cos 2x\cancel{\sin x}}{2\cancel{\sin x}\cos x}$

$\lim_{x\to0}\frac{1}{t} = \lim_{x\to0}\left(\cos x+\frac{\cos 2x}{2\cos x}\right) = 1+\frac{1}{2} = \frac{3}{2}$ $\therefore t = \frac{2}{3}$

方法2：原式 $= \lim_{x\to0}\frac{\sin 2x}{x}\cdot\frac{x}{\sin 3x} = \lim_{x\to0}\left(2\frac{\sin 2x}{2x}\right)\lim_{x\to0}\left(\frac{1}{3}\frac{3x}{\sin 3x}\right)$

$= \frac{2}{3}$

3.7 $\lim_{x\to0}\frac{1-\cos x}{x}$

方法：对 $1-\cos x$ 使用降幂公式

原式 $= \lim_{x\to0}\frac{2\sin^2\frac{x}{2}}{x} = \lim_{x\to0}\frac{\sin\frac{x}{2}\sin\frac{x}{2}}{x/2} = \lim_{x\to0}\sin\frac{x}{2}\cdot\lim_{x\to0}\frac{\sin\frac{x}{2}}{x/2}$

$= 0\cdot1 = 0$

3.7 $\lim_{x\to0}\frac{\cos x-1}{2x^2}$ 方法：对 $\cos x-1$ 使用降幂公式

$= \lim_{x\to0}\frac{-2\sin^2\frac{x}{2}}{2x^2} = \lim_{x\to0}\frac{-\sin^2\frac{x}{2}}{4\cdot\left(\frac{x}{2}\right)^2} = \frac{-1}{4}$

3.8 $\lim_{x\to0}\frac{\sin^2 x}{x} = \lim_{x\to0}\left[(\sin x)\left(\frac{\sin x}{x}\right)\right]$

$= \lim_{x\to0}\sin x\cdot\lim_{x\to0}\frac{\sin x}{x} = 0\cdot1 = 0$

3.9 $\lim_{x\to0}\frac{\sin x}{\sqrt[3]{x}}$

$= \lim_{x\to0}\left(\frac{\sin x}{x}\cdot\frac{\sqrt[3]{x^2}}{1}\right) = 1\cdot0 = 0$

3.10 $\lim_{x\to\pi}\frac{\sin x}{x} = \left(\lim_{x\to\pi}\sin x\right)\left(\lim_{x\to\pi}\frac{1}{x}\right)$

$= 0\cdot\frac{1}{\pi} = 0$

Exercise 2: For any $\varepsilon > 0$, there exists $\delta > 0$, when $|x-0| < \delta$, there is $\left|\frac{f(x)}{x}-1\right| < \varepsilon$, find $\lim_{x\to0} f(x)$.

$\forall\varepsilon>0$, $\exists\delta>0$, $C=0$, $L=1$ $\quad \lim_{x\to0}\left(\frac{f(x)}{x}\right)=L=1$

$\because \lim_{x\to0} x=0 \quad \therefore \lim_{x\to0}\left(\frac{f(x)}{x}\right)=1$ is an indeterminate form $\frac{0}{0}$

$\therefore \lim_{x\to0} f(x)=0$

总结：1. $\frac{0}{0}$ 不定型 → 分子有理化 $\quad \lim_{x\to0}\frac{\sqrt{x+5}-\sqrt{5}}{x}$

2. $\Delta x\to0$ 型的 $\frac{0}{0}$ 不定式 $\quad \lim_{\Delta x\to0}\frac{(x+\Delta x)^3-x^3}{\Delta x} \Longrightarrow$ 展开

$\lim_{\Delta x\to0}\frac{\sqrt{x+\Delta x}-\sqrt{x}}{\Delta x} \Longrightarrow$ 分子有理化

3. $\frac{0}{0}$ 不定型求参数 $\quad \lim_{x\to2}\frac{x^2+ax+b}{x-2}=3$ 求 a, b

4. $\frac{\sin x}{x}$ 型 $\quad \lim_{x\to0}\frac{\sin^2 x}{x}$

5. $\cos x-1$ 降幂公式 $\quad \lim_{x\to0}\frac{\cos x-1}{2x^2}$

1.1 If $\lim_{x\to0}\frac{\tan ax}{\sin 3x} = -2$, find a.

原式 $= \lim_{x\to0}\left(\frac{a}{a}\frac{\tan ax}{x}\cdot\frac{3}{3}\frac{x}{\sin 3x}\right)$

$= \lim_{x\to0}\left[a\left(\frac{\tan ax}{ax}\right)\frac{1}{3}\left(\frac{3x}{\sin 3x}\right)\right]$

$= \frac{a}{3}\lim_{x\to0}\frac{\tan ax}{ax}\lim_{x\to0}\frac{3x}{\sin 3x}$

$= \frac{a}{3} = -2 \Longrightarrow a=-6$